A Simple Jump to Default Model

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Abstract

A simple jump to default model is used to illustrate preference and position dependent derivatives pricing in incomplete markets, with the emphasis on how to make *systematic* trading decisions based on the model.

KEYWORDS: DERIVATIVES, INCOMPLETE MARKET

I Introduction

In this pedagogical article, I study derivatives pricing and trading under a simple jump to default model, in which the stock price follows a geometric Brownian motion with constant drift and volatility. During any time interval dt, the stock price can jump to zero (default) with probability ζdt. To express it mathematically, the stock movement is governed by a jump diffusion model

$$ds = s[(\nu + \zeta)\, dt + \sigma\, dB - dQ] \tag{1}$$

where dB is the standard Brownian motion, and dQ is the Possion jump term that has the probability ζdt of being one and $(1 - \zeta dt)$ of being zero. Note that the three model parameters ν, σ and ζ are regarded as known constants.

Imagine you are playing a trading game with four types of instruments in it: (i) the stock, (ii) risky bonds, (iii) vanilla European call options on the stock, and (iv) convertible bonds. This game offers a little bit of everything, and the game is over once the default occurs. The stock can be traded continuously; whereas the other types of instruments are considered as derivatives that are not trading continuously. This setting is a good approximation to the situation where the bid-ask spread on the stock is always nearly zero (liquid), and the ones on derivatives are usually large (illiquid). The game shows the current price p of a derivative, it then asks you for your trading decision, whether to buy or sell, and how many. Note after the initial derivative trade, the game does not guarantee that you will be able to trade the same instrument again before its maturity. Since there are four types of instruments in the game, there are quite a few combinations of trading them together, some of which will be discussed later in this article.

Jump to default models are not new, it started with the work of Samuelson and Merton (see Chapter 9 of [11]), other variants of the model can be found in papers [1, 4, 5, 8, 13] and references therein. But none of these papers discusses the trading issue addressed in this article. Most jump to default models in the literature are more complicated than the one presented here. Instead of being realistic, I want the model to be as simple as possible for pedagogical reasons, but still has the following two features: (i) it has a continuous trading component, so some sort of dynamic hedging is allowed; (ii) it has to be an incomplete market model, otherwise there is no trading game to speak of. In the spirit of making matters as simple as possible, both the interest rate and dividend rate are set to zero, as the case of deterministic interest and dividend rates causes little technical difficulty and does not change any of the qualitative results presented below. Furthermore, other idealized assumptions also apply to this game, such as allowing short selling and leverage (borrowing), and no transaction costs. This simple jump to default model, which is an one-parameter extension of the classical Black-Scholes (BS) model,[1] offers plenty insights into derivatives pricing and trading in incomplete markets.

The material in this article is *not* self-contained. This pedagogical article can be regarded as a sequel to [15], so you must read [15] to understand all the basic concepts involved before proceeding. Glancing over first part of [16] will also be helpful, but not required to understand this article.

II Pure Stock Investment Problem

In this section, the only instrument you are allowed to trade is the stock. You may wonder what this has anything to do with derivatives trading, which is the focus of this article. The answer, which will be shown later, is that in incomplete markets, derivatives pricing depends on the solution of the pure stock investment problem.

The pure stock investment problem is about portfolio optimization in continuous time, *i.e.*, finding a strategy that maximizes the expected utility of the terminal wealth. In this article, I use the exponential utility function

$$U(w) = -\frac{1}{\gamma}\exp(-\gamma w) \tag{2}$$

where $\gamma > 0$ is the risk aversion parameter. One approach to find the optimal strategy is to apply the standard stochastic control theory (stochastic dynamic programming), *i.e.*, the HJB equation; much like in calculus where you set the first order derivative of a function to zero to find a maximum. If you know the basics of the stochastic control theory, then what comes next is not new. However, if you are completely new to this subject, then hold your nose and go through the next few paragraphs, because you do not need to have a thorough knowledge of the topic to do computations.

The method used to solve a portfolio optimization problem is relevant to derivatives pricing, because in the new paradigm to be presented shortly, derivatives are priced in the context of portfolio optimization. The simple pure stock investment problem offers a glimpse of the procedure involved in deriving the option pricing equations.

Assume the amount of money invested in the stock is π, which is a variable you can control at any moment by buying or selling the stock, then the number of shares n_0 in the portfolio is simply π/s. The change of wealth dw during a small time interval dt is given by the following budget equation

$$dw = n_0\, ds = \pi\,\frac{ds}{s} = \pi[(\nu+\zeta)\,dt + \sigma\, dB - dQ] \tag{3}$$

[1] The other one-parameter incomplete market extension of the BS model that I know of is to include proportional transaction costs [6, 14], which turns out to be a harder problem than the simple jump to default model here [16].

Let the value function be $J(t, w)$, which is the maximized expected utility at the current time t based on the current wealth w,[2] then the infinitesimal change of J during a small time interval is

$$d\,J(t,w) = J_t\,dt + J_w\,dw + \frac{1}{2}J_{ww}(dw)^2 + [J(t,w-\pi) - J(t,w)]dQ \tag{4}$$

where subscripts denote partial derivatives. The necessary condition for optimality is the HJB equation $\sup_\pi E[dJ] = 0$. After substituting the expression for dw, it becomes

$$\sup_\pi \{J_t(t,w) + \pi(\nu+\zeta)J_w(t,w) + \frac{1}{2}\pi^2\sigma^2 J_{ww}(t,w) + \zeta[J(t,w-\pi) - J(t,w)]\} = 0 \tag{5}$$

The symbol $\sup_\pi$ means that the derivative of the left hand side with respect to π is zero, which produces

$$(\nu+\zeta)J_w(t,w) + \bar\pi\sigma^2 J_{ww}(t,w) - \zeta J_w(t,w-\bar\pi) = 0 \tag{6}$$

where $\bar\pi$ is the optimal investment amount.

One advantage of using the exponential utility function is that the wealth variable can be separated out, *i.e.*, $J(t, w)$ is of the form

$$J(t,w) = -\frac{1}{\gamma}\exp[-\gamma w - \gamma\bar\phi(t)] \tag{7}$$

Substituting this expression for J into (6) leads to[3]

$$(\nu+\zeta) - \gamma\bar\pi\sigma^2 - \zeta\exp(\gamma\bar\pi) = 0 \tag{8}$$

For a given ν, (8) is a transcendental equation for $\bar\pi$, which always has a unique root, as the left hand side is a monotone decreasing function in $\bar\pi$ that goes from $+\infty$ to $-\infty$. Therefore there is a one-to-one relation between the drift parameter ν and the optimal investment amount $\bar\pi$. In reality, the drift coefficient ν is very difficult to estimate accurately. Since ν is unknowable in practice, it is your belief that really matters. Taking a directional bet $\bar\pi$ implies having a particular view on ν. The practical advantage of viewing ν as a function of $\bar\pi$ (see Fig. 1) through (8) is that $\bar\pi$ is directly under your control. It is clear from the equation as well as the figure that making a large bullish bet requires an exponentially large positive drift due to the default risk. Notice that directional neutral ($\bar\pi = 0$) means $\nu = 0$, and vice versa.

Substituting expression (7) into (5) gives the equation for $\bar\phi$

$$\bar\phi_t + \bar\pi(\nu+\zeta) - \frac{1}{2}\gamma\bar\pi^2\sigma^2 - \frac{\zeta}{\gamma}[\exp(\gamma\bar\pi) - 1] = 0 \tag{9}$$

The final condition is $\bar\phi(T) = 0$, where T is the investment horizon. It is trivial to solve this equation with the solution being

$$\bar\phi(t) = \{\bar\pi(\nu+\zeta) - \frac{1}{2}\gamma\bar\pi^2\sigma^2 - \frac{\zeta}{\gamma}[\exp(\gamma\bar\pi) - 1]\}(T-t) \tag{10}$$

where ν and $\bar\pi$ are related by (8). It is not difficult to show that $\bar\phi$ is always positive. The financial interpretation for $\bar\phi$ is that it is the CEPL of playing the stock trading game, *i.e.*, you are indifferent between the choices of applying the optimal stock trading strategy or receiving a lump sum $\bar\phi$ (but forfeit trading the stock).

To summarize, at any given time, the optimal strategy for the pure stock investment problem is to have $\bar n_0 := \bar\pi/s$ shares in the portfolio. The CEPL of this optimal strategy is given by (10).

[2] Actually J should also be a function of the stock price s in general. It turns out in the end that this problem is s independent, so the s variable is left out from the onset.

[3] A similar equation for a much general model under the power utility function is given in [9].

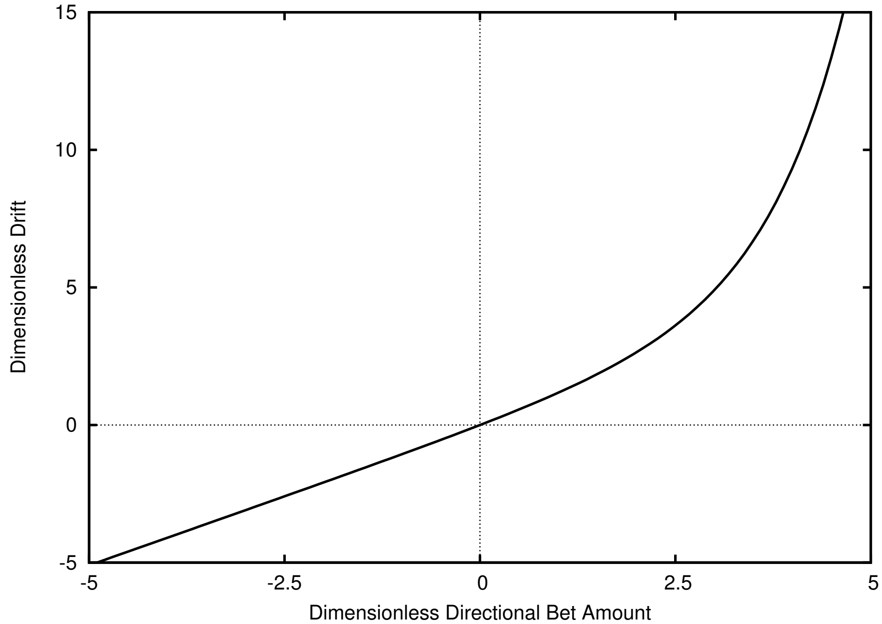

Figure 1: The horizontal axis is the dimensionless directional bet amount $\gamma\bar{\pi}$; the vertical axis is the dimensionless drift ν/σ^2. The dimensionless default intensity here is $\zeta/\sigma^2 = 0.1$.

III SJD-DOPE

In this section, I present the derivative pricing equations under the simple jump to default model (1). Let $\vec{n}$ be a vector representing a portfolio of European derivatives, where the component n_i $(i = 1, 2, \cdots, N)$ is the position of the ith instrument with maturity date T_i and payoff function $f^i(T_i, s)$. Suppose currently you do not have any derivative position, what is the most amount you want to pay to take over the portfolio $\vec{n}$?

This amount h (you are being paid if $h < 0$) is called the portfolio indifference price for position $\vec{n}$, as you are indifferent between the lump sum h and position $\vec{n}$. For the simple jump to default model, $h(t, s)$ satisfies the following nonlinear PDE, whose origin will be discussed later in this section,

$$h_t + \frac{1}{2}\sigma^2 s^2 h_{ss} - \frac{1}{2}\gamma\sigma^2 s^2 (n_0^h + h_s)^2 + [\sigma^2 + \zeta \exp(\gamma\bar{\pi})] s (n_0^h + h_s) + \sum_{i=1}^{N} n_i f^i(T_i, s)\delta(t - T_i) = 0 \tag{11}$$

where γ is the risk aversion parameter and the optimal hedging shares of the stock n_0^h can be computed from solving the transcendental equation

$$\gamma s (n_0^h + h_s)\sigma^2 + \zeta \exp(\gamma\bar{\pi})\{\exp[\gamma s (n_0^h + h_s) - \gamma(\Delta h + s h_s)] - 1\} = 0 \tag{12}$$

The quantity Δh in (12) is simply $h(t, 0) - h(t, s)$. In this trading game, all four types of instruments, *i.e.*, the stock, risky bonds, vanilla European calls and convertible bonds, become worthless when the default occurs, so $h(t, 0) = 0$, which means $\Delta h = -h$. It is easy to show that for any given $h(t, s)$, equation (12) for n_0^h always has a unique root. With all terms in (11) known, h can be propagated backward, at least numerically, from its final condition $h(T, s) = 0$, where T is the investment horizon that is larger than any T_i. Note that whenever a maturity is crossed, h jumps by the amount of the payoff function times its position size.

Equation (12) says that the number of optimal dynamic hedging shares n_0^h depends on h alone, which is meaningful only at the portfolio level. Since h satisfies a nonlinear PDE, hedging the whole portfolio is not a linear sum of hedging each instrument in it. Of course, when $\vec{n} = \vec{0}$, h is zero, so is n_0^h. Note that the total number of shares of the stock in the portfolio is $\hat{n}_0 = \bar{n}_0 + n_0^h$, where $\bar{n}_0 := \bar{\pi}/s$ is the number of shares corresponding to the optimal directional bet.

The fair value of a derivative is defined to be a model output such that you are a buyer/seller if the market price is lower/higher than it, and you stay put when the market price equals your fair value. Once h and n_0^h are solved, the fair value $f^i(t,s)$ of the ith instrument satisfies the linear PDE[4]

$$f_t^i + \frac{1}{2}\sigma^2 s^2 f_{ss}^i + [\zeta \exp(\gamma\bar{\pi}) - \gamma s(n_0^h + h_s)\sigma^2](\Delta f^i + s f_s^i) = 0 \tag{13}$$

with the final condition at T_i being the known payoff function $f^i(T_i, s)$. Once again $\Delta f := f(t,0) - f(t,s) = -f(t,s)$, because all derivatives in the game's trading universe become worthless after the default.

Equation (13) says that the fair value is in general position dependent, through the position effect term $(n_0^h + h_s)$, which implies that the model market is incomplete. Whenever necessary, I will make the position dependency notationally explicit by writing h and f^i as $h(t,s|\vec{n})$ and $f^i(t,s|\vec{n})$. In the following two special situations the position effect term $(n_0^h + h_s)$ is identically zero: (i) when $\zeta = 0$ (cf. (12)), as expected the fair value equation (13) reduces to the well-known BS equation that is position independent; (ii) when the derivative position is empty, *i.e.*, $\vec{n} = \vec{0}$ (implies $h = 0$), again (13) reduces to the well-known BS equation, but with an upward adjusted interest rate, which is a known result of Merton (see Chapter 9 of [11]).

The pair of PDEs (11) and (13) is called SJD-DOPE, which is the simple jump to default model version of the dynamic option pricing equations. The SJD-DOPE will be solved numerically later on, with the emphasis on studying the position dependent effect. The position dependency of the SJD-DOPE offers a natural and systematic way to trade derivatives [15].

It is clear that the SJD-DOPE depends on the stock drift parameter ν through the optimal directional bet amount $\bar{\pi}$, which proves the earlier assertion that the solution of the pure stock investment problem affects derivatives valuation. This contrasts the BS complete market models ($\zeta = 0$) where ν has no bearing on option pricing. In this simple jump to default model, the influence of the drift comes into the equations through the modified default intensity $\bar{\zeta} := \zeta \exp(\gamma\bar{\pi})$. Therefore for the purpose of pricing derivatives, the three-parameter model (ν, σ, ζ) becomes effectively a two-parameter one $(\sigma, \bar{\zeta})$. In other words, studying the effect of the optimal directional bet $\bar{\pi}$ is the same as studying the effect of default intensity ζ, with a bullish bet increasing $\bar{\zeta}$ and a bearish one decreasing it. Without loss of generality, I will only consider the directional neutral case $\bar{\pi} = 0$ in the rest of this article.

Ideas Behind the SJD-DOPE Derivation

I now address the question of where the SJD-DOPE comes from. The answer is that it comes from solving a portfolio optimization problem with a twist. The mathematical tool involved is the stochastic control theory, a glimpse of which has been provided in Section II. The details of the SJD-DOPE derivation are

[4] A similar PDE is given in Section 14.5.3 of [12], the major difference is that I derived an explicit expression for an unspecified function $g(t,s)$ there.

presented in Appendix, as it follows a well-defined procedure, which is somewhat tedious. However, the ideas behind the derivation are explained here.

Suppose in addition to the stock price, the market price of a derivative instrument is also being exogenously specified as a process, what will you do? Note this is no longer a derivative pricing problem, it becomes a portfolio optimization problem. In complete markets, the solution is trivial: you buy infinite amount if the derivative market price is lower than its BS value, conversely you sell infinite amount when it is higher. When the two numbers equal, it does not matter what you do, as long as you use the delta hedging scheme to replicate the derivative payoff function. In incomplete markets, however, it is possible to specify both price processes (the underlying asset and one of its derivatives) without causing arbitrages. In general, solving such a portfolio optimization problem using the stochastic control theory is not a trivial exercise [3, 10].

In a standard portfolio optimization problem, the market price process is the input, and the optimal trading strategy is the output. The twist here, and it is an important one, is to turn this around and ask the question of what the market price process should be in order for a given trading strategy to be optimal. I now show you why this twist offers a natural solution to the derivatives pricing problem.

Recall that derivatives are regarded as illiquid assets in the trading universe, *i.e.*, they cannot be traded continuously. Since the game may not offer you another chance to trade the same instrument, the only trading strategy available to you, whenever the game gives you a chance to trade a derivative, is to change your current position and expect to hold the position to maturity. The key insight is that your position remains constant after the initial trade. Because the derivative is illiquid, you really cannot observe its market price process (think of wide bid-ask spread); however, you are free to imagine a fictitious market price process. Now the question becomes under what fictitious price process will a constant position strategy be optimal.

At this point, apply the standard machinery of the stochastic control theory, which says that the necessary condition of optimality is the HJB equation. Now go through some algebraic steps, the final outcome is that the fictitious market price must satisfy equation (13). When the current observed market price p agrees with the solution of (13), then you have found a fictitious market price process that starts with p and ends with $f^i(T_i, s)$ under which the position $\vec{n}$ remains optimal.[5] Simply put, your current fair value process is the necessary condition under which your current portfolio remains optimal.

When the market price and your current fair value do not agree, then the necessary condition of optimality is violated. The only logical conclusion is that your current position is not optimal. Thus you need to adjust your position until the post-trade based fair value agrees with the market price. By pricing derivatives in the context of portfolio optimization, you have obtained a natural and systematic way of trading derivatives. This new method of deriving the option pricing equation by embedding it in a portfolio optimization problem is called dynamic derivation. It recovers the well-known BS equation (cf. (13) with $\zeta = 0$) in a complete market (see also Section 3.2 of [16]).

IV Trading Risky Bonds

Risky bonds of a single maturity are studied in this section. There are two situations to cover: (i) dynamic hedging using the stock is allowed, and (ii) naked bond position without hedging. The reason that the

[5] Technically speaking, this is only the necessary condition for optimality, you need to go through the verification step to ensure that sufficient conditions are satisfied as well.

stock can be used to hedge risky bonds is that they share a common risk factor—default.

Let me consider the hedging situation (i) first. Since the payoff function of the unit face value bond is s independent, it is not difficult to see that the s dependency in (11) and (12) drops out because the optimal stock hedging amount $\pi^h := n_0^h s$ is also independent of s. Thus PDE (11) becomes an ODE for $h(t)$

$$\gamma h_t - \frac{1}{2}\sigma^2(\gamma\pi^h)^2 + (\sigma^2 + \zeta)(\gamma\pi^h) = 0 \tag{14}$$

where the dimensionless optimal stock hedging amount $\gamma\pi^h$ satisfies the transcendental equation

$$\sigma^2(\gamma\pi^h) + \zeta[\exp(\gamma h + \gamma\pi^h) - 1] = 0 \tag{15}$$

The final condition for h is

$$h(T) = n \tag{16}$$

where I have assumed that there are n units of the T maturity risky bond in the portfolio. Note that the impulse delta function in (11) has been converted into the final condition (16). The fair value PDE (13) becomes the ODE

$$f_t + [(\gamma\pi^h)\sigma^2 - \zeta]f = 0 \tag{17}$$

with the final condition being $f(T) = 1$. This set of ODEs does not seem to have a simple analytical solution, but numerical computation is easy to carry out.

In the no hedging situation (ii), it is easy to write down explicit expressions for the portfolio indifference price h and the fair value f (see [15]):

$$\begin{aligned} \gamma h(t) &= -\ln[1 - \exp(-\zeta\bar{t}) + \exp(-\zeta\bar{t} - \gamma n)] && (18) \\ f(t) &= \frac{\exp(-\zeta\bar{t} - \gamma n)}{1 - \exp(-\zeta\bar{t}) + \exp(-\zeta\bar{t} - \gamma n)} && (19) \end{aligned}$$

where $\bar{t} := T - t$ is the time to maturity. Note that the default probability for the risky bond during the time interval $[t, T]$ is simply $1 - \exp(-\zeta\bar{t})$. Let me point out that the two expressions (18) and (19) do not respectively satisfy the ODEs (14) and (17), because the ODEs are derived under the assumption that dynamic stock hedging is allowed.

I first study the effect of the bond maturity. The fair value of a risky bond can be converted into an implied spread over the riskless rate; since the riskless rate is zero here, the implied spread is the same as the implied yield on the bond, which is given by $y_i := -\ln f/\bar{t}$. It is easy to check that when $n = 0$, the dimensionless implied spread y_i/ζ is one for both cases (hedging and nonhedging). In Fig. 2, the dimensionless implied spread y_i/ζ is plotted against the dimensionless time to maturity $\zeta\bar{t}$ for two different dimensionless position sizes γn and three different dimensionless default intensities ζ/σ^2. Notice that the dotted curves within each row are the same, as it is easy to verify from (19) that the implied spread is independent of ζ/σ^2. It is clear from the figure that the hedging and nonhedging scenarios exhibit the same qualitative behaviors. The spread shrinks under a short position, because the fair value of the risky bond is raised relative to the no position case; similarly, the spread widens under a long position, because the fair value is lowered. Under both long and short positions, the dimensionless spread approaches one as the time to maturity lengthens. For a given short position, the spread for the nonhedging scenario

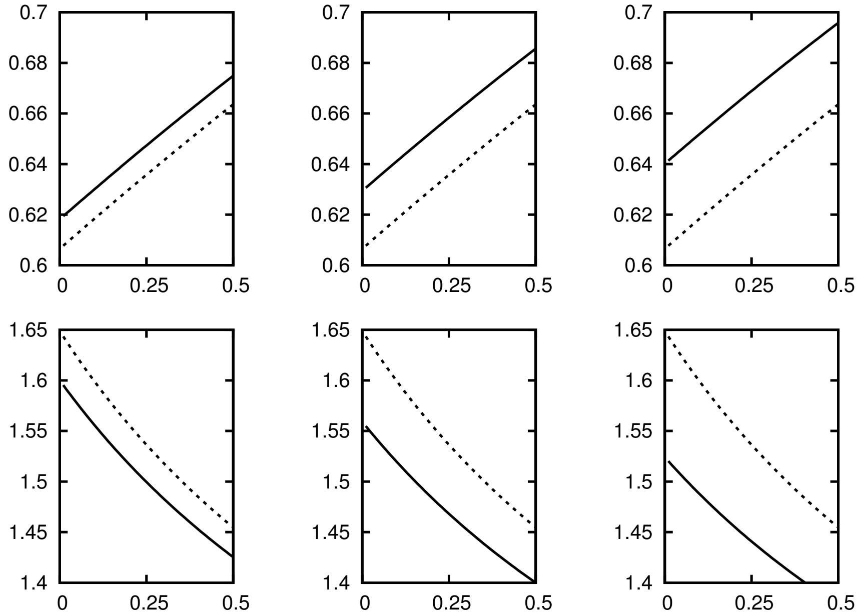

Figure 2: The horizontal axes are the dimensionless time to maturity $\zeta\bar{t}$; the vertical axes are the dimensionless implied spread y_i/ζ. The upper row is for $\gamma n = -0.5$; the bottom row is for $\gamma n = 0.5$. The left, middle and right columns are for $\zeta/\sigma^2 = 0.05$, 0.1, and 0.15, respectively. The solid curves are for the hedging scenario, whereas the dotted ones are for the nonhedging scenario.

is smaller, because it demands a higher bond price to short the same amount as the hedging scenario; similar reasoning shows that the spread for the nonhedging scenario is larger under a given long position. This can also be seen from the optimal CEPL plot Fig. 3, where the dotted curves always lie about the corresponding solid ones because the nonhedging scenario demands a better price concession to establish the same position.

Under the hedging scenario, the optimal hedging ratio $-\pi^h/(nf)$ is of great interest, where $\pi^h := n_0^h s$ is the amount money invested in the stock and nf is the amount invested in the risky bond. The normalized optimal hedging ratio is plotted in Fig. 4. It is obvious that the optimal hedging ratio defined with a minus sign should always be positive, as you short the stock to hedge a long bond position, and vice versa. One striking feature of Fig. 4 is that the optimal hedging ratio is almost independent of the bond maturity, which means you need more shares to hedge a short maturity bond, as the bond fair value is a decreasing function of time to maturity. The optimal hedging ratio itself will be sensitive to the default intensity, but after normalizing it with the dimensionless default intensity ζ/σ^2, it becomes a slow varying function of ζ/σ^2 as changing ζ/σ^2 by a factor of three (from left to right columns) does not alter the normalized optimal hedging ratio by that much. It is also clear from the figure that for the same absolute size, the long bond position (bottom row) requires a higher hedging ratio.

I now investigate the effect of position size, which is needed to determine the optimal trading size. The most important things to know with respect to making rational trading decisions on the risky bond are its quote and reserve price curves, which are plotted in the top row of Fig. 5. Notice that I have assumed that the pre-trade portfolio has no derivatives in it, otherwise the current fair value for hedging and nonhedging cases will not be the same. The formulas of the quote and reserve price curves are given by $q(m) = f(*|m)$

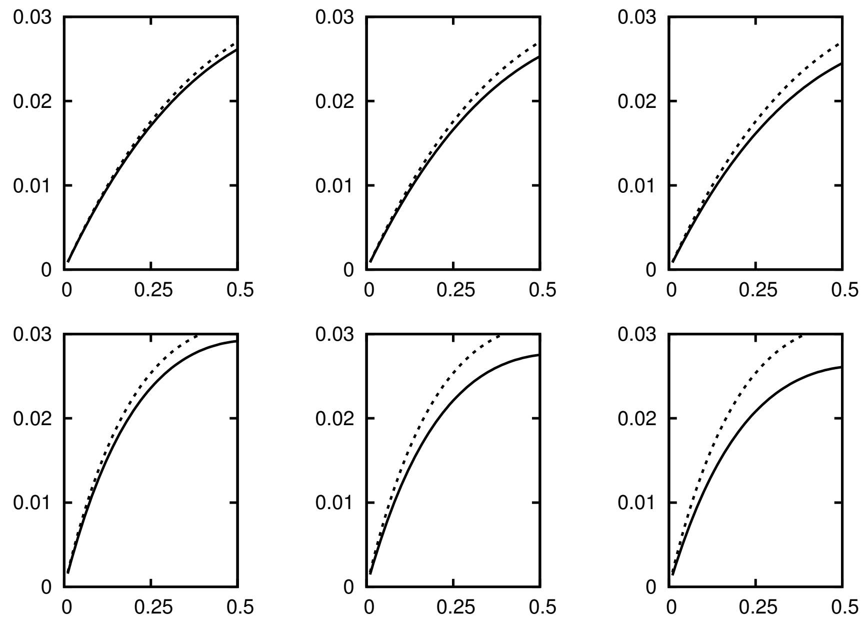

Figure 3: Similar to Fig. 2, except that the vertical axes are the dimensionless optimal CEPL $\gamma h(*|n) - \gamma n f(*|n)$.

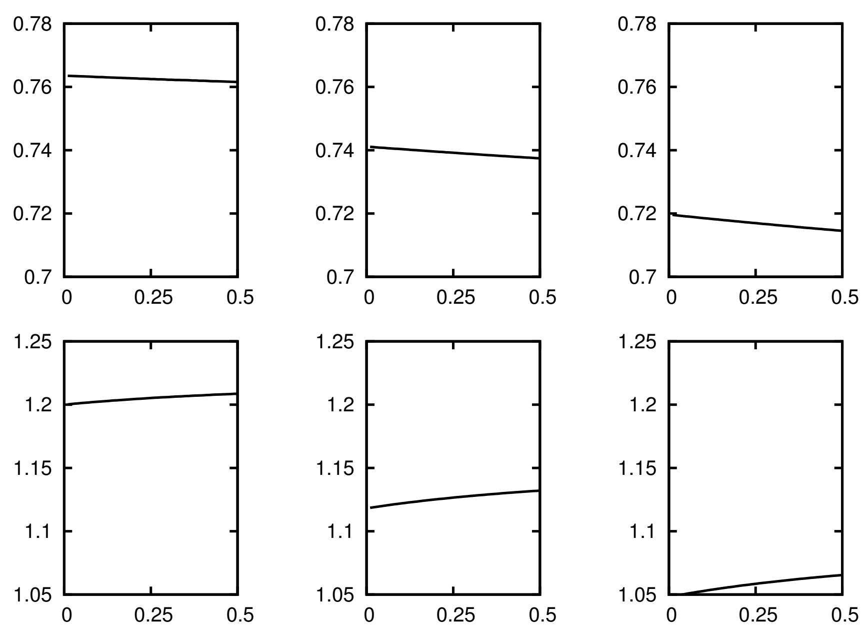

Figure 4: Similar to Fig. 2, except that the vertical axes are the normalized optimal hedging ratio $-(\sigma^2 n_0^h s)/(\zeta n f)$.

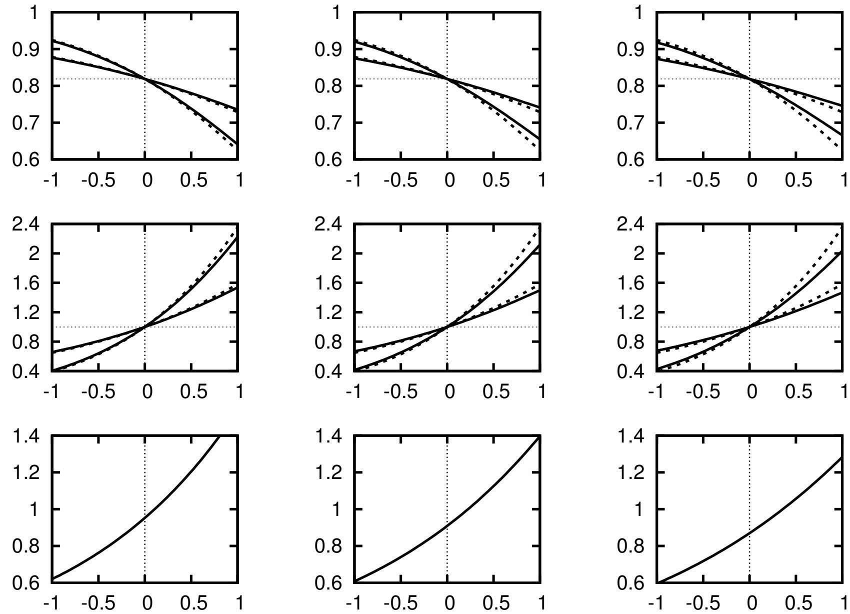

Figure 5: The horizontal axes are the dimensionless trading size; the vertical axes of the top row are the trading price; the vertical axes of the middle row are the corresponding dimensionless implied spread y_i/ζ; the vertical axes of the bottom row are the normalized optimal hedging ratio $-(\sigma^2 n_0^h s)/(\zeta n f)$. The left, middle and right columns are for $\zeta/\sigma^2 = 0.05$, 0.1, and 0.15, respectively. The solid curves are for the hedging scenario, whereas the dotted ones are for the nonhedging scenario.

and $r(m) = h(*|m)/m$, respectively [15]. As expected, the quote and reserve price curves of the nonhedging scenario have larger negative slopes. The middle row is similar to the top row, except that both the quote and reserve prices have been converted to their respective implied spreads. The curves in the middle row have positive slopes, because a higher price means a lower spread.

For all panels in Fig. 5, the dimensionless time to maturity $\zeta\bar{t}$ is fixed at 0.2. For a given default intensity ζ, the three different columns correspond to three different volatilities σ, with left column having the largest σ (smallest ζ/σ^2). Note that the dotted curves of different columns within each row are the same, as the nonhedging case does not depend on σ. Because the deviations of the solid curves from the corresponding dotted ones are smallest in the left column, it is easy to conclude that you hedge less when σ is large.

The normalized optimal hedging ratio is plotted in the bottom row of Fig. 5. It is not too sensitive to σ, which means that the optimal hedging ratio $-(n_0^h s)/(nf)$ is almost inversely proportional to σ^2 under a given ζ. Thus a larger σ means smaller hedging ratio (less hedging), which is consistent with the previous observation. The optimal hedging ratio is an increasing function of the trading size, or position size here, with a large short position having the lowest ratio and a large long position have a highest ratio, which is also consistent with the result of Fig. 4.

The optimal CEPL curves corresponding to the middle column of Fig. 5 ($\zeta/\sigma^2 = 0.1$) is plotted in Fig. 6. Because the quote price curve $q(m)$ is a monotone decreasing function, there is a one-to-one mapping between the trading price and the optimal trading size, which maps the horizontal axes of the two panels in the figure to each other. The left half of the left panel corresponds to the right half of the

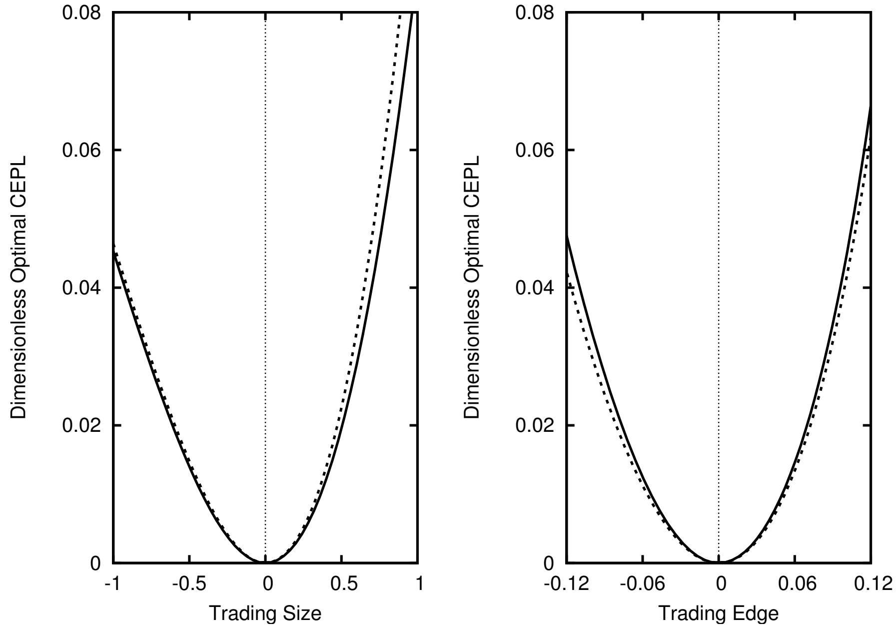

Figure 6: The horizontal axis for the left panel is the dimensionless trading size γm, the one for the right panel is the trading edge $p - q(0)$, where p is the trading price and $q(0)$ is the current fair value; the vertical axis is the dimensionless optimal CEPL $\gamma h(*|m) - \gamma n f(*|m)$. The solid curves are for the hedging scenario, whereas the dotted ones are for the nonhedging scenario.

right panel, because by convention selling ($m < 0$) is associated with positive trading edges. As mentioned before, for a given trading size (left panel), the CEPL for the nonhedging scenario is higher because it demands a larger price concession. However, for a given trading price, or trading edge, the nonhedging case has a lower CEPL because its corresponding optimal trading size is smaller. Another intuitive way to understand this is that you would choose not to hedge if the hedging CEPL were lower than that of the nonhedging one.

V Trading European Calls

The only instruments you can trade in this section are European calls, and the underlying stock that is used for dynamic hedging. The emphasis of the study is on how fair values are affected by the underlying position.

I first investigate how the fair value of the at-the-money call (relative strike $k/s = 1$, dimensionless time to maturity $\sigma^2 \bar{t} = 0.1$) changes with respect to its position in the portfolio. This information is contained in the quote price curve of the option in Fig. 7, assuming the pre-trade position is an empty portfolio. Recall that the sell arbitrage price is defined to be a price beyond which you want to sell unlimited quantities, and the buy arbitrage price is the one below which you want to buy unlimited quantities. It can be inferred from Fig. 7 that $f(*|m)/s$ approaches one as $m \to -\infty$, which means that the sell arbitrage price for the option is simply s. Obviously if the call were selling at or above s, you could sell it and buy the stock with one-to-one ratio to arbitrage. It is clear from the same figure that the fair value approaches its BS value (without default) in the $m \to +\infty$ limit, which means that the buy arbitrage price for the option is the BS

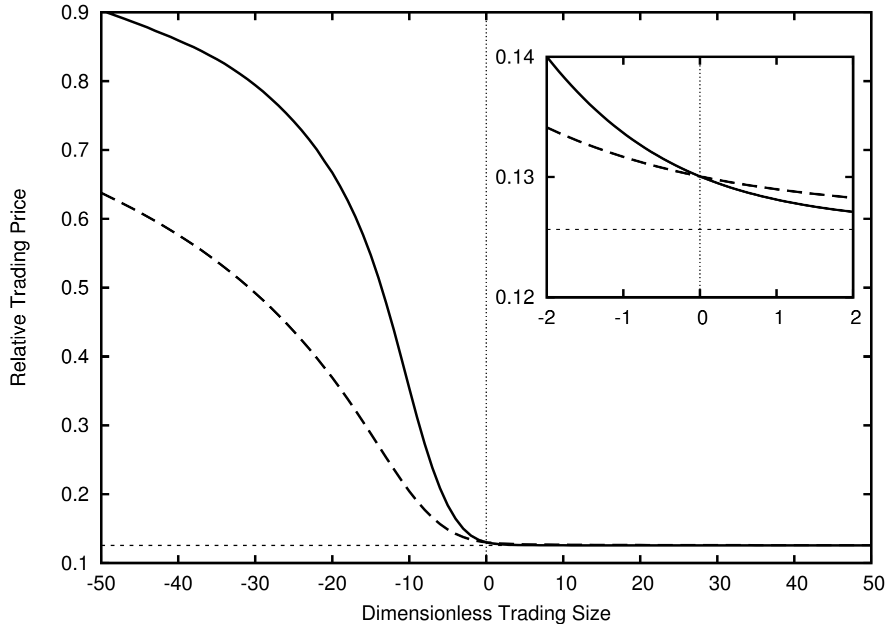

Figure 7: The horizontal axis is the dimensionless trading size γm; the vertical axis is the relative trading price p/s. The solid curve is for quote price, and the dashed one is for reserve price. The dotted horizontal line is the position independent BS value with no default. The inset is a blow-up of the center region.

value, not zero! Let us look into what happens if you buy the call at its BS price and use the regular delta hedging scheme to hedge the position. If the default does not occur during the lifetime of the option, then you break even. However, if the default does occur, you gain sf_s with f_s being the shares shorted, but lose f. Since f is a convex function in s, $sf_s - f > 0$.[6] Thus you cannot lose whether default occurs or not. Indeed you would want to buy an infinite amount if the call were to trade at or below its BS value. Note that the arbitrage argument does not work if the stock volatility is stochastic.

Any European call value can be converted into an implied volatility by the BS formula (no default). Since your personal fair value of a given call is position dependent, so is your implied volatility surface $\sigma^i(k, \bar{t}|\vec{n})$, where k is the strike and $\bar{t}$ is the time to maturity. In the rest of this section, the main focus will be on how the implied volatility surface changes with respect to the underlying option position $\vec{n}$.

Let us first consider what the implied volatility surface looks like when there is no position in the portfolio (zero-option-portfolio). Three different time to maturity slices of $\sigma^i(k, \bar{t}|\vec{0})$ for three different default intensity are plotted in Fig. 8. There are no surprises from this figure. The implied volatility surface is negatively skew due to the downside jump risk. Furthermore, the skew is stronger when ζ is larger, which is also intuitive.

I now discuss how the implied volatility surface changes when there are options in the portfolio. The natural quantity to focus on is the relative percentage change of the implied volatility with respect to that of the zero-option-portfolio

$$100\,\frac{\sigma^i(*|\vec{n}) - \sigma^i(*|\vec{0})}{\sigma^i(*|\vec{0})} \tag{20}$$

[6] Merton (see Chapter 8 of [11]) proved that when the stock return does not depend on the stock level, which is the case here, call option values are convex functions of the stock price.

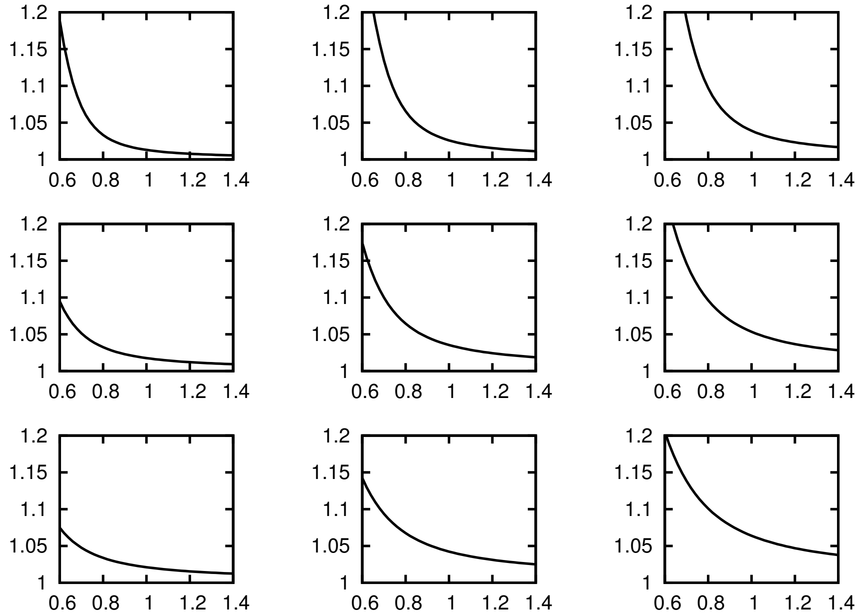

Figure 8: The horizontal axis is the relative strike price k/s; the vertical axis is the relative implied volatility σ^i/σ. The top, middle and bottom rows are for $\sigma^2\bar{t} = 0.05$, 0.1 and 0.15, respectively. The left, middle and right columns are for $\zeta/\sigma^2 = 0.05$, 0.1 and 0.15, respectively.

I will investigate three cases next where $\vec{n}$ represents, respectively, (i) an at-the-money call, (ii) a vertical call spread (long the lower strike and short the upper strike with 1 : 1 ratio) with relative strikes "0.75 : 1.25", and (iii) a butterfly (long the wing strikes each once, and short the middle strike twice) with relative strikes "0.75 : 1.0 : 1.25", with the dimensionless time to maturity being fixed at $\sigma^2\bar{t} = 0.1$ for all three cases.

The first case is to study the impact of the at-the-money call option, which is plotted in Fig. 9. It is clear that the long position suppresses implied volatilities, whereas the short position elevates them. This result will be proved analytically later in Section VI.

The second case is the vertical call spread position, which is plotted in Fig. 10. A long vertical call spread position (long the lower strike call and short the upper strike call) depresses the implied volatility of the lower strike due to the long position, whereas the implied volatility of the upper strike is elevated due to the short position, so the slope is positive. For similar reasons, negative slope curves are associated with the short position. Despite positive slopes for the curves in Fig. 10 corresponding to the long vertical call spread position, the overall skew for σ^i remains negative. Because the relative change of the implied volatility in this example is not large enough to overcome the negative skew of the zero-option-position; it simply lessens the original negative skew.

The third case is the butterfly position, which is plotted in Fig. 11. The long position has negative gamma in the middle, so the curves look like those of a short the at-the-money call position Fig. 9; similar analogy exists for the short butterfly position. The magnitude of the change here is much smaller because a butterfly position is balanced comparing to an outright call position. Notice that in the middle row of Fig. 11, which has the same maturity as the underlying position, the curves in the middle (near at-the-money) are concave and convex for the long and short positions, respectively. The concaveness associated with the long butterfly position is probably due to the largest elevation of the implied volatility in the

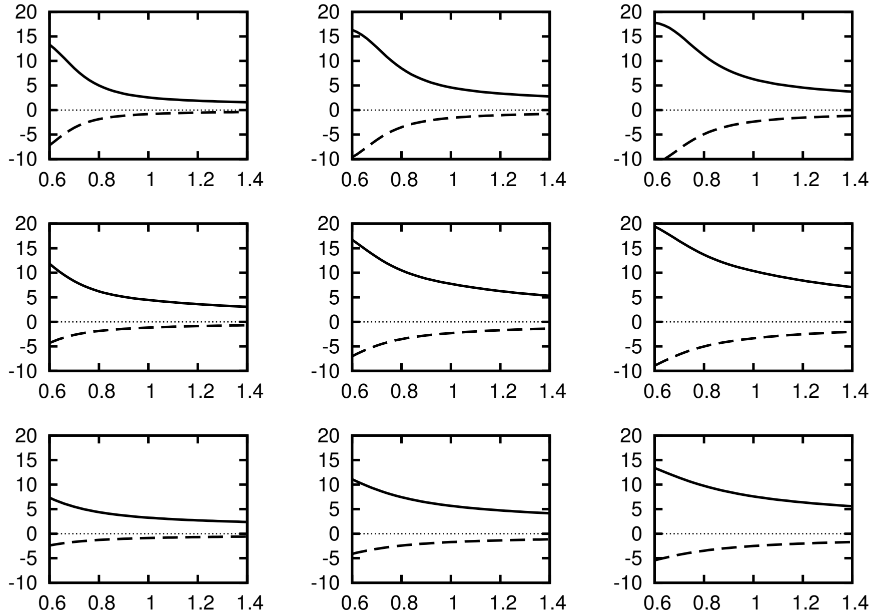

Figure 9: Relative percentage changes of implied volatilities from the ones in Fig. 8 for the at-the-money call position with dimensionless size $|\gamma ns| = 2$. The solid curve is for the short position and the dashed one is for the long position.

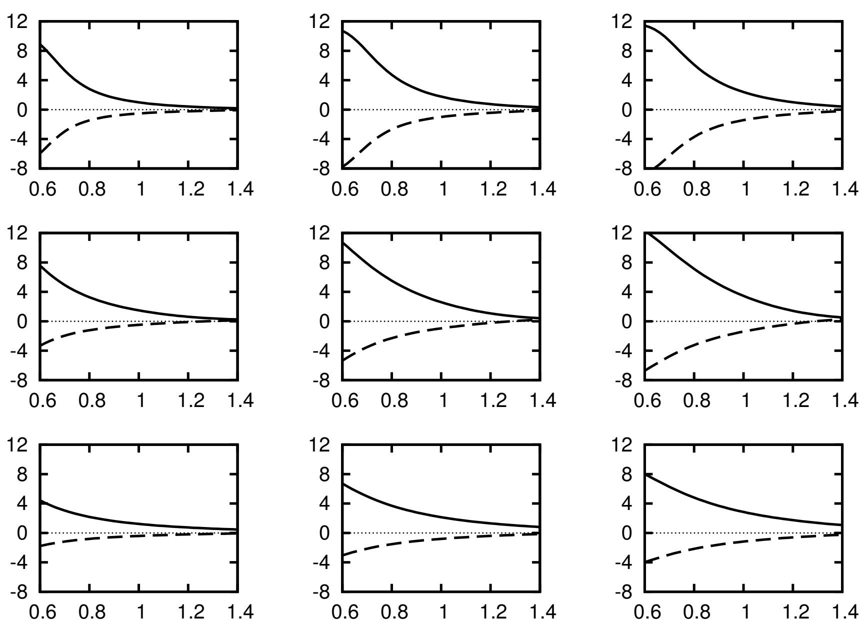

Figure 10: Similar to Fig. 9, except that the underlying position is the "0.75 : 1.25" vertical call spread. The solid curve is for the short position $\gamma ns = -2$ and the dashed one is for the long position $\gamma ns = 2$.

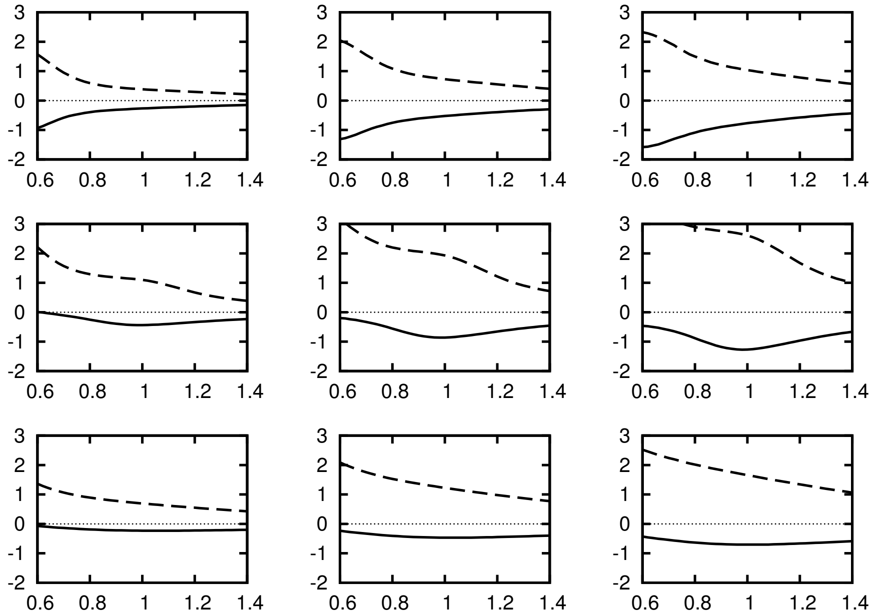

Figure 11: Similar to Fig. 9, except that the underlying position is the "0.75 : 1.0 : 1.25" butterfly. The solid curve is for the short position $\gamma ns = -2$ and the dashed one is for the long position $\gamma ns = 2$.

middle from the short call of the middle strike relative to those that close to the wings. Similar logic works for the convex shape.

The three figures, Fig. 9, Fig. 10 and Fig. 11, exhibit some common features: (i) the position effect is stronger when the default intensity is larger, which is intuitive, as the zero default limit is the BS complete market model that has no position effect; (ii) the position impact is less in terms of implied volatility for longer maturities; (iii) the implied volatilities on the downside strikes are very sensitive to the position effect in this simple jump to default model.

I continue to investigate the position effect on the implied volatility surface, but with large positions this time. What plotted below are actual implied volatility surfaces, not the relative percentage change (20). The implied volatility surface for the empty portfolio is plotted in Fig. 12, where the three curves in the right panel correspond to the ones of the middle column of Fig. 8. For comparison purposes, the scale is set to be the same as the ones on two other figures, which I now describe. The implied volatility surface for the large long vertical spread position is plotted in Fig. 13. Notice that the position effect can cause positive skews for at-the-money options, *i.e.*, $\frac{d\sigma^i}{dk}|_{k=s} > 0$ (cf. the right panel). The implied volatility surface for the large long butterfly position is plotted in Fig. 14, where the position effect makes the surface frown in the middle (cf. the solid curve in the right panel), *i.e.*, $\frac{d^2\sigma^i}{dk^2}|_{k=s} < 0$.

Compare to the empty portfolio case, the at-the-money implied volatility for the maturity $\sigma^2\bar{t} = 0.1$ has more than doubled. This is somewhat surprising especially for the butterfly position, as it is considered to be balanced. I think this is due to the unusual feature of this model where a large long call position cannot suppress the relative implied volatility, as it has a lower bound of one instead of zero. Because of this, the net effect of the combination of large long and short positions is that the overall implied volatilities are raised.

In summary, you now have seen conclusive evidence that in an incomplete model, the implied volatility

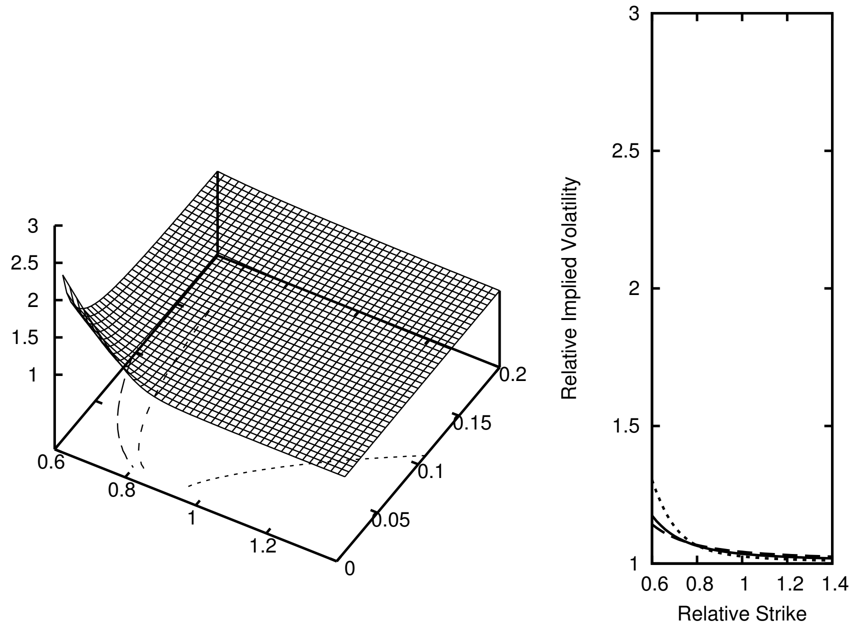

Figure 12: The underlying portfolio is empty. The left panel is the implied volatility surface σ^i/σ. The three contours on the bottom are for level sets 1.02, 1.10 and 1.18. The right panel shows three different time to maturity slices at $\sigma^2\bar{t} = 0.05$ (dotted), 0.1 (solid) and 0.15 (dashed). The dimensionless default intensity ζ/σ^2 is 0.1.

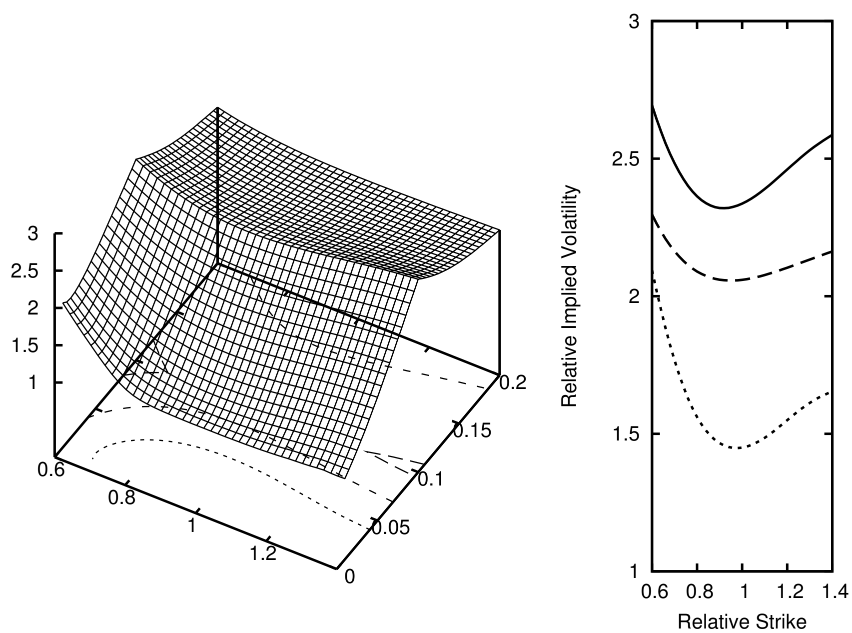

Figure 13: Similar to Fig. 12, except that the underlying portfolio is long the "0.75:1.25" vertical call spread at $\sigma^2\bar{t} = 0.1$ with dimensionless position size $\gamma ns = 50$. The contours on the bottom are for level sets 1.5, 2.0 and 2.5.

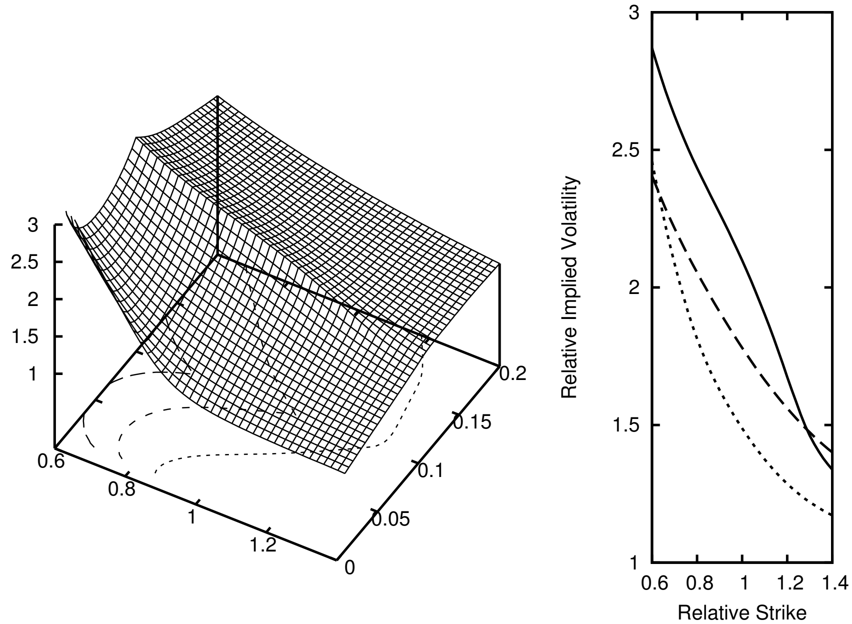

Figure 14: Similar to Fig. 12, except that the underlying portfolio is long the "0.75:1.0:1.25" butterfly at $\sigma^2\bar{t} = 0.1$ with dimensionless position size $\gamma ns = 50$. The three contours on the bottom are for level sets 1.5, 2.0 and 2.5.

surface changes its shape when the underlying position changes,[7] *i.e.*, the implied volatility surface can take on a variety of shapes depending on the underlying position. The position effect in incomplete markets offers a very plausible explanation on why implied volatility surfaces in the real world fluctuate.

VI Trading Everything

In this section, I examine portfolios that contain both risky bonds and European calls. First let me discuss the dimensional (units) issue. So far I have been presenting all results in dimensionless units, as there are natural ways to make various dimensional quantities dimensionless. For example, in Section IV on trading risky bonds, the natural way to make time t dimensionless is to multiply it with default intensity ζ; in Section V on trading European calls, $\sigma^2 t$ is the natural dimensionless time. But the situation here is not so clear-cut, because of this I will use both dimensional and dimensionless numbers. The following parameters are used in the numerical computation: the default intensity is one percent per year ($\zeta = 0.01$); and the annual stock volatility is 31.62% ($\sigma^2 = 0.1$). Without loss of generality, the current stock price is set to one ($s = 1$).

Suppose the two-year at-the-money call is trading at $p^c = 0.1817$ (relative implied volatility being 1.027), and the ten-year unit face value risky bond is trading at $p^b = 0.8881$ (implied spread being 1.09%), what would you do? assuming that your initial portfolio is empty. Let me investigate one thing at a time first. If you cannot trade the risky bond, then the CEPL for trading n_c units of the call at the fixed price p^c is $h(*|n_c, 0) - n_c p^c$, which is plotted in the left panel of Fig. 15. It is clear that the optimal trading size

[7]The position effects for other incomplete models, such as stochastic volatility models with or without transaction costs, are presented in [16].

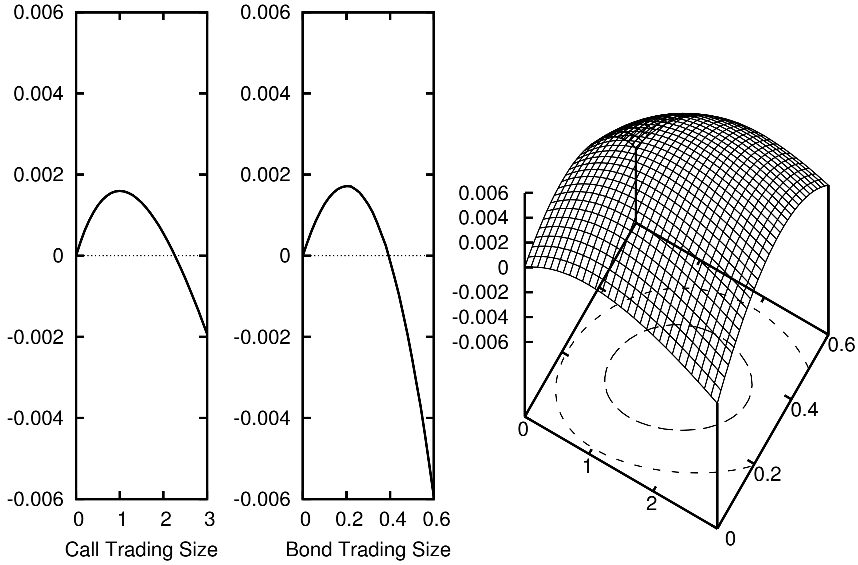

Figure 15: The vertical axes are the dimensionless CEPL $\gamma\Upsilon$. The contours on the bottom of the right panel are for level sets 0.002 and 0.004. The left two panels are cross sections of the surface plot along the two bottom axes.

occurs at $\gamma n_c = 1.0$. If you cannot trade the European call, then the CEPL for trading n_b units of the bond at the fixed price p^b is $h(*|0, n_b) - n_b p^b$, which is plotted in the middle panel of Fig. 15. It is clear that the optimal trading size occurs at $\gamma n_b = 0.2$. When there are no trading constraints, the optimal trading sizes for the call and the bond are found by numerically maximizing the CEPL $h(*|n_c, n_b) - n_c p^c - n_b p^b$, which is plotted in the right panel of Fig. 15. The solution is $\gamma n_c = 1.482$ and $\gamma n_b = 0.2962$. Since $h(*|\vec{n})$ is global concave function in the position argument, the optimal trading size problem under a given set of trading prices always has a unique answer as long as the given prices do not allow arbitrage.

Notice when the trading restrictions are lifted, the optimal trading sizes for both the call and the bond have increased (from 1.0 to 1.482 for the call and from 0.2 to 0.2962 for the bond), which implies that these two instruments has complimentary risk characteristics, in a sense I now make more precise.

It will be apparent in a minute that the sign of the quantity $v^i := \Delta f^i + s f^i_s = -f^i + f^i_x$, where $x := \ln s$, plays an important role. PDE (13) for the fair value can be rewritten as

$$f^i_t + \frac{1}{2}\sigma^2 v^i_x + Q_1 v^i = 0 \tag{21}$$

where Q_1 is defined as

$$Q_1 := \zeta \exp(\gamma\bar{\pi}) - \gamma s (n^h_0 + h_s)\sigma^2 \tag{22}$$

Using the definition for v^i and (21), it is easy to show that v^i satisfies the PDE

$$v^i_t + (\frac{1}{2}\sigma^2 + Q_1)v^i_x + \frac{1}{2}\sigma^2 v^i_{xx} + (Q_{1x} - Q_1)v^i = 0 \tag{23}$$

This is a diffusion type linear homogeneous PDE, applying the well-known Feynmann-Kac formula leads to the conclusion that if $v^i(T_i, s)$ has a single sign, then $v^i(t, s)$ will have the same sign. If the ith instrument is a European call, then $v^i(t,s) > 0$, as $v^i(T_i,s) \geq 0$; but if the ith instrument is a risky bond, then $v^i(t,s) < 0$, as $v^i(T_i,s) < 0$. Note unlike the case in Section IV where fair values for risky bonds are stock price independent, as soon as there are calls in the portfolio, $f^i_s \neq 0$ for any risky bond due to the position effect term $(n^h_0 + h_s)$.

I now investigate how the fair value f^i of the ith instrument changes if the underlying position is perturbed along the direction $\vec{e}_j$ of the jth instrument, where $\vec{e}_j$ is a vector of zeros except that the jth entry is one. This change is characterized by the quantity f^{ij}, which is defined as

$$f^{ij}(t,s|\vec{n}) := \lim_{m\to 0} \frac{1}{m}\Big[f^i(t,s|\vec{n}+m\vec{e}_j) - f^i(t,s|\vec{n})\Big] = \frac{\partial f^i}{\partial n_j} \tag{24}$$

Taking partial derivatives with respect to n_j on PDE (13) and the hedging equation (12) leads to

$$f^{ij}_t + Q_1 s f^{ij}_s + \frac{1}{2}\sigma^2 s^2 f^{ij}_{ss} - Q_1 f^{ij} - Q_2 v^i v^j = 0 \tag{25}$$

where Q_2 is defined as

$$Q_2 := \frac{\gamma\sigma^2\zeta\exp(\gamma\bar{\pi})\exp(\gamma n^h_0 s - \gamma\Delta h)}{\sigma^2 + \zeta\exp(\gamma\bar{\pi})\exp(\gamma n^h_0 s - \gamma\Delta h)} > 0 \tag{26}$$

Note that the derivation involves the use of the tangent relation $\partial h/\partial n_j = f^j$. The final condition on f^{ij} is obviously zero, as the payoff function is position independent. To summarize, f^{ij} satisfies a linear inhomogeneous diffusion type PDE with zero final condition. Observe that the source term of the PDE $-Q_2 v^i v^j$ is always negative if both the ith and the jth instruments belong to the same type (both calls or both bonds), however, the source term is positive if they belong to different types (one call and one bond).

Applying the well-known Feynman-Kac formula with zero final condition to (24) leads to the immediate conclusion that $f^{ij} < 0$ if the ith and the jth instruments belong to the same type, and $f^{ij} > 0$ if they belong to different types. The financial interpretation is as follows: For European calls, buying/selling a call lowers/raises fair values for all calls regardless of strikes and maturities, or equivalently buying/selling a call suppresses/elevates implied volatilities for all calls. For risky bonds, buying/selling a bond lowers/raises fair values of all bonds regardless of maturities, or equivalently buying/selling a bond widens/shrinks implied yield spreads for all risky bonds. These conclusions are intuitive, they are the consequence of risk aversion in incomplete markets.

Two instruments are said to have the same risk characteristics if buying/selling one always lowers/raises the fair value of the other; similarly they are said to have complimentary risk characteristics if buying/selling one always raises/lowers the fair value of the other. Note putting two instruments of complimentary risk characteristics together in a portfolio has the effect of hedging each other. European calls and risky bonds have complimentary risk characteristics, as buying one will make you want to buy more of the other, which explains the earlier result that when trading restrictions are lifted, the final local equilibrium state contains more calls and bonds.

Trading a Convertible Bond

Now enlarge the trading universe by including a five-year unit face value convertible bond (CB) with conversion ratio 0.8. The conversion price, *i.e.*, the strike price of the embedded equity option is $1.0/0.8 = 1.25$. In other words, the convertible bond can be viewed as a subportfolio containing one unit of five-year risky bond and 0.8 units of five-year European call with the strike being 1.25.

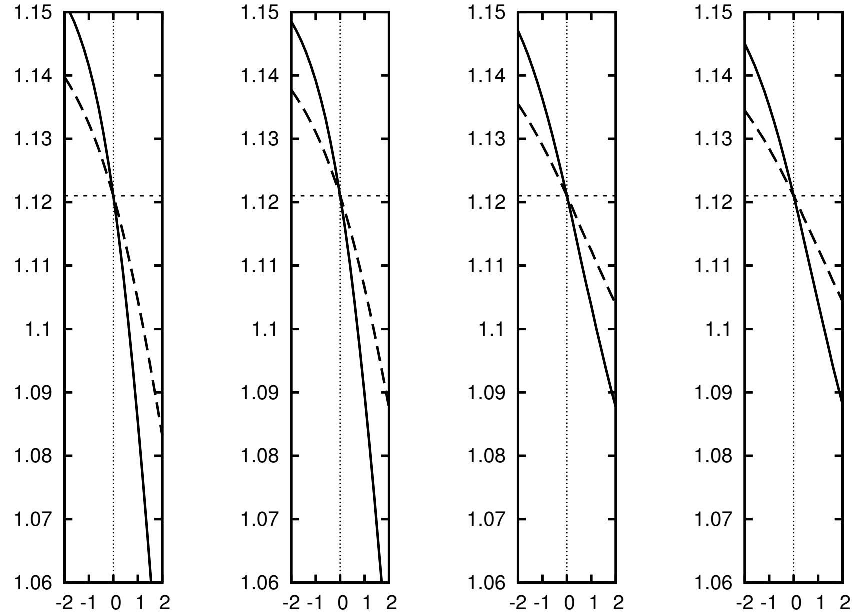

Figure 16: The horizontal axis is the dimensionless trading size γm for the CB. The vertical axis is the trading price. The solid curves are for the quote price and the dashed ones are for the reserve price. The horizontal dotted lines represent the current fair value for the CB (1.121). The panels from left to right are for (i) no static hedging, (ii) static hedging with the call only, (iii) static hedging with the bond only, and (iv) static hedging with both instruments.

Recall that the current portfolio (pre-trade position) contains $\gamma n_c = 1.482$ units of the two-year at-the-money call, and $\gamma n_b = 0.2962$ units of the ten-year risky bond. In addition you are short $\gamma n_0^h = -0.8628$ shares of the stock to hedge the pre-trade position. The fair value for the convertible bond based on the pre-trade position is 1.121.

The focus next is on how to make rational trading decisions with respect to this convertible bond under various static hedging conditions,[8] which are (i) no static hedging, (ii) static hedging with the call only at price 0.1817, (iii) static hedging with the bond only at price 0.8881, (iv) static hedging with both instruments at the stated price. Note that dynamic hedging using the stock is allowed in all four cases.

The quote and reserve price curves of the CB under the four static hedging cases are plotted in Fig. 16. As usual, the quote and reserve price curves are flatter when more instruments are available for static hedging, *i.e.*, when the market is more complete. Note if the hedging instruments were a five-year bond and a five-year call with strike 1.25, then the quote and reserve price curves for the CB would have been flat lines, because the CB payoff function can be statically spanned by the new hedging instruments, so arbitrage pricing is applicable. In this example, the ten-year bond is a much better hedging instrument than the two-year call, as there is a big difference between the second and third panel in the figure. Furthermore when both hedging instruments are present, the impact of the two-year call is small, especially in the buying segment of the curves (see the third and forth panels). This fact can also be seen from the optimal CEPL plots in Fig. 17, as the dash dotted and the solid curves are very close. Earlier comments on Fig. 6 are also applicable to the current optimal CEPL figure.

[8]See Section XII of [15] for basic concepts and formulas with regard to static hedging.

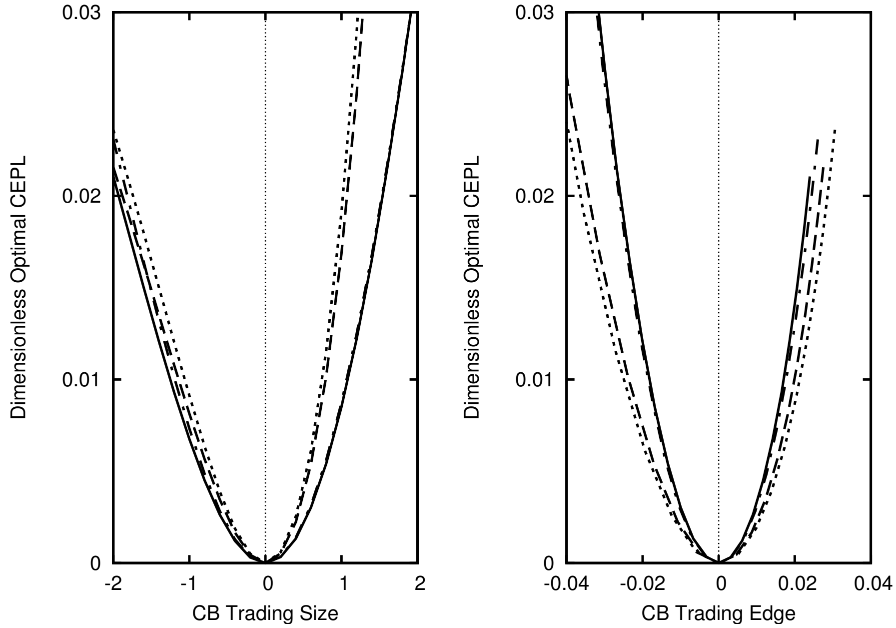

Figure 17: The horizontal axis for the left panel is the dimensionless trading size γm, the one for the right panel is the trading edge $p - q(0)$, where p is the trading price and $q(0)$ is the current fair value 1.121; the vertical axis is the dimensionless optimal CEPL $\gamma \Upsilon_o$. The lines are for (i) no static hedging (dotted), (ii) static hedging with the call only (dashed), (iii) static hedging with the bond only (dash dotted), and (iv) static hedging with both instruments (solid).

I now examine the hedging quantities of various hedging instruments, which are computed endogenously from the model without any guesswork. The number of extra shares of the stock used for hedging the portfolio is plotted in the left panel of Fig. 18, by extra I mean it is the optimal hedging shares based on the post-trade position minus the shares used to hedge the pre-trade position (-0.8628). Because the nonlinear position effect are not strong for small positions, the curves in the left panel are close to straight lines, which would have been the case if the model market were complete. Obviously you need to short the stock to hedge a long CB position, and vice versa. The right panel shows that you need to short the ten-year bond to hedge a long CB position, and vice versa, which is also intuitive. However, the middle panel is interesting. It says that you need to long the two-year call to hedge a long CB position, even at the presence of the ten-year bond (the solid curve in the middle panel). It turns out that in this example the bond component risk of the CB outweighs its call component risk. Since the maturities of the five-year CB and the ten-year bond do not quite match, the system requires you long the two-year call to hedge the residual bond component risk in the CB, bearing in mind that calls and bonds have complimentary risk characteristics. You may wonder what happens when the ten-year bond is replaced by the five-year bond for static hedging. Such a static hedging scenario is plotted in Fig. 19. As expected, when the hedging bond maturity matches that of the CB, the optimal static hedge requires you to short the two-year call for a long CB position, and vice versa, see the solid curve in the middle panel.

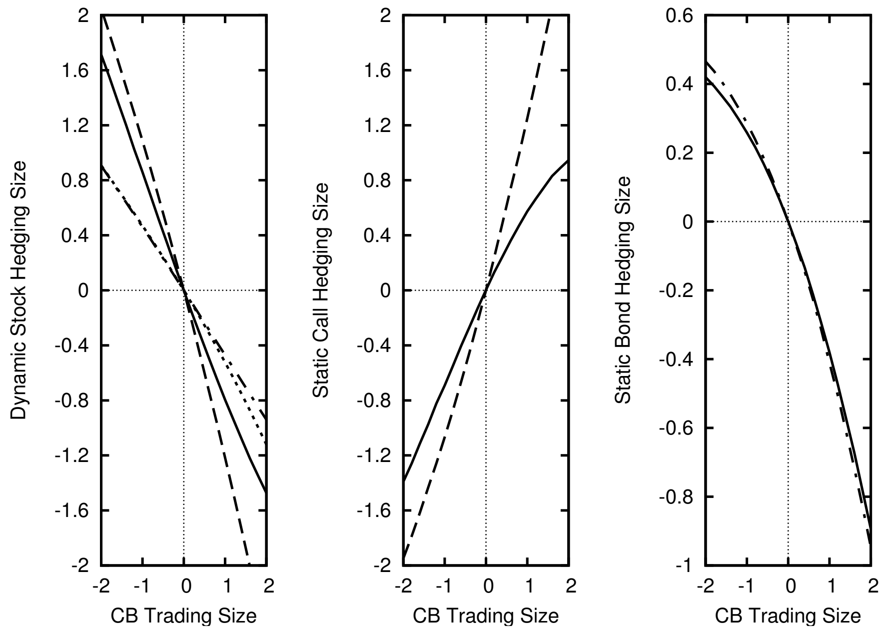

Figure 18: The horizontal axis is the dimensionless trading size γm for the CB. The vertical axis for the left panel is the number of extra shares of the stock used to dynamically hedge the portfolio at the current state (t, s); the vertical axis for the middle panel is the number of calls used in static hedging; the vertical axis for the right panel is the number of bonds used in static hedging. The lines are for (i) no static hedging (dotted), (ii) static hedging with the call only (dashed), (iii) static hedging with the bond only (dash dotted), and (iv) static hedging with both instruments (solid).

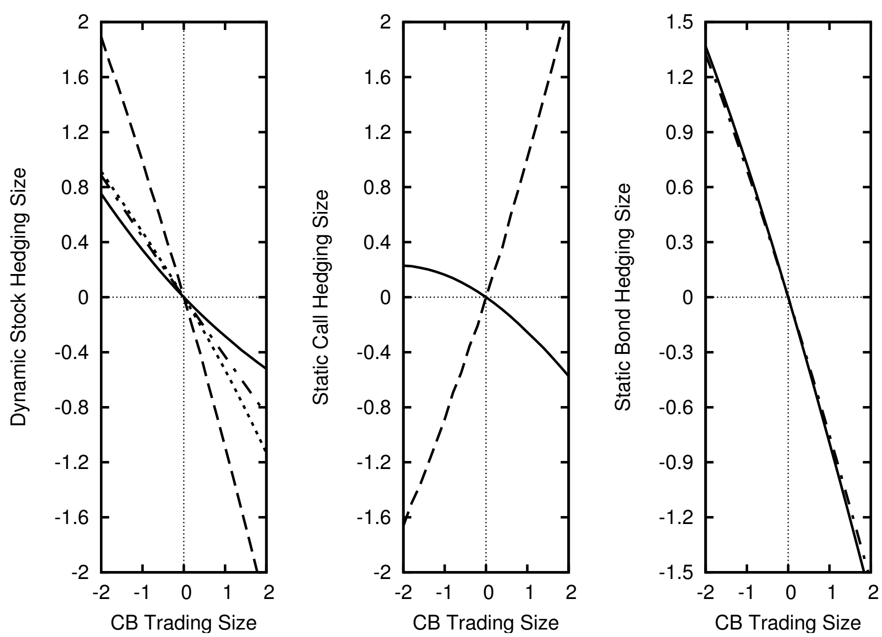

Figure 19: Similar to Fig. 18, except for the different static hedging instruments and hedging prices, which are: the two-year at-the-money call (same as before) trading at 0.1853, and the five-year bond (replacing the ten-year bond) trading at 0.9512.

VII The Magical BS Formula

Suppose the market price of the longest maturity risky bond is specified as the following continuous-time process

$$db = u(t)b\,dt - b\,dQ \tag{27}$$

where $u(t)$ is a *deterministic* positive function. Furthermore you are allowed to trade this bond continuously without any transaction cost. So now you have two instruments to do dynamic hedging, *i.e.*, the stock and the longest maturity risky bond. In this situation, the market is complete, which is intuitive, as the two risk sources dB and dQ can now be eliminated through dynamic tradings of the stock and the bond.

In such a complete market, all other instruments can be priced using preference-free arbitrage arguments. The hedging strategy for other risky bonds is simply to short the same amount of the longest maturity bond. It is easy to see that prices of other risky bonds must satisfy (27) as well to avoid arbitrages. For a T_i maturity European call option f^i, a portfolio containing one call, $-f^i_s$ shares of the stock, and the amount $f^i - sf^i_s$ invested in risky bonds of the same maturity is immune to both the stock movement risk as well as the default risk. Applying Ito's lemma leads to the PDE

$$f^i_t - u(t)sf^i_s + \frac{1}{2}\sigma^2 s^2 f^i_{ss} + u(t)f^i = 0 \tag{28}$$

which has the solution (for a European call)

$$f^i(t, s, T_i, k) = sN(d_1) - z^i kN\left(d_1 - \sigma\sqrt{T_i - t}\right) \tag{29}$$

where $N(x)$ is the cumulative normal distribution function, k is the strike price, z^i is the market price of the T_i maturity risky bond, and $d_1 := \left[\ln(s) - \ln(z^i k) + \frac{1}{2}\sigma^2(T_i - t)\right]/\sigma\sqrt{T_i - t}$. This is simply the well-known BS formula except that the discount factor is replaced by the price of the risky bond [2].

Formula (29) is derived based on the arbitrage argument of a complete market. There is no doubt about the mathematical validity of the derivation. The issue is about modeling. Is the bid-ask spread on the long maturity risky bond almost always near zero? Does the empirical observation of the market price process satisfy (27)? If the answer to either one of these two questions is negative, then it is inappropriate to complete the market by make such an *aggressive* assumption, *i.e.*, being able to continuously trade a security with an exogenously specified price process. Therefore it is reasonable to question the usefulness of formula (29) whose derivation is based on conditions that are rarely satisfied in reality.

Actually the validity of the BS formula (29) extends well beyond its complete market based derivation, hence the magic. I now show that (29) is also valid under the SJD-DOPE when there are no call positions in the portfolio. With only bonds and the stock in the portfolio, both the portfolio indifference price h and the optimal stock hedging amount $\pi^h := n^h_0 s$ are independent of s, which is the situation discussed in Section IV. Hence the position effect term $s(n^h_0 + h_s)$ in (13) is only a function of t, so (13) can be written as

$$f^i_t - \tilde{u}(t)sf^i_s + \frac{1}{2}\sigma^2 s^2 f^i_{ss} + \tilde{u}(t)f^i = 0 \tag{30}$$

where $\tilde{u}(t)$ can be computed given an underlying bond position. Let the market price of the T_i maturity risky bond be z^i, you can always trade with the market to establish a local equilibrium in which your fair value of the T_i bond based on the post-trade position is also z^i. Having done that, your fair value for any

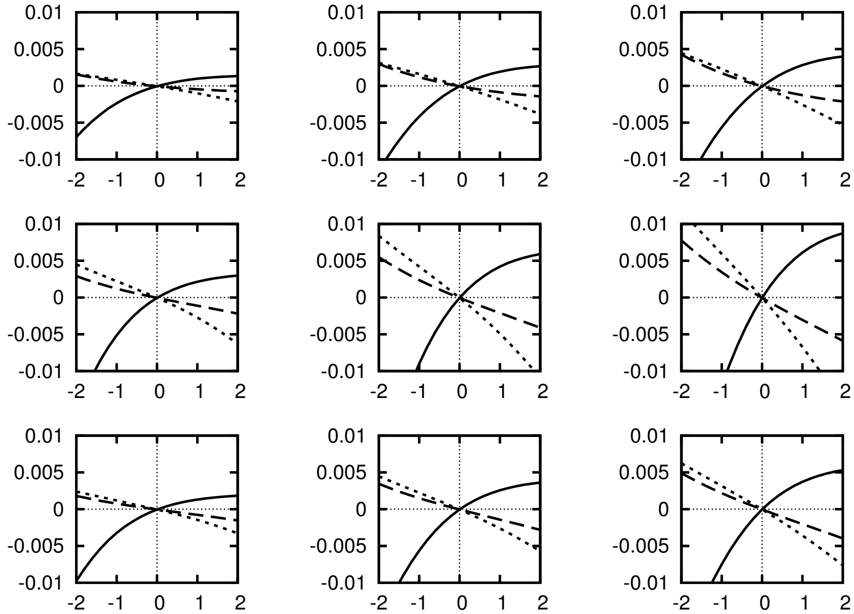

Figure 20: The vertical axes are relative errors between (29) and that of the SJD-DOPE for at-the-money calls with three different time to maturities: $\sigma^2\bar{t} = 0.05$ (top row), 0.1 (middle row), and 0.15 (bottom row). The horizontal axes are the dimensionless position size $\gamma|\vec{n}|$ for the underlying position, where $\vec{n}$ represents: (i) at-the-money call (solid), (ii) "0.75:1.25" vertical call spread (dashed), (iii) "0.75:1.0:1.25" butterfly (dotted). The time to maturity for all three positions is fixed at $\sigma^2\bar{t} = 0.1$. The left, middle and right columns are for $\zeta/\sigma^2 = 0.05$, 0.1, and 0.15, respectively.

European call is then given by (29), because the fair value of the T_i risky bond satisfies the same equation (30) (with derivatives with respect to s being zero).

Even with small call positions, formula (29) is expected to do well because it has the same qualitative inventory control feature of the SJD-DOPE solution. This conclusion is based on the fact that f^i in (29) is a decreasing function of z^i. The argument goes like this: buying a call raises the fair value of the T_i risky bond (since they have complimentary risk characteristics), so f^i according to (29) is lowered (because z^i has increased), which agrees with that of the SJD-DOPE (two calls have the same risk characteristics). The other scenarios (selling a call, buying a bond, and selling a bond) can be analyzed analogously. In each scenario, the qualitative outcome of (29) agrees with that of the SJD-DOPE. To see how good an approximation formula (29) is to that of the SJD-DOPE, relative errors of at-the-money calls under various position and position sizes are plotted in Fig. 20. The overall conclusion is that for moderate position sizes, the magical formula (29) has only about a few percent relative errors. Let me comment that the test positions in Fig. 20 do not contain bonds. Adding bonds to the underlying position can noticeably change curves in the figure. Needless to say all curves pass through the origin.

The BS formula has been battle tested by practitioners for many years. It is reassuring that SJD-DOPE confirms its usefulness. The BS formula is often used as an interpolation device [7], in the sense that parameters in the formula are replaced by market prices of relevant instruments. Such ad hoc procedures usually work well in reality, but theoretical justifications for them are often murky. The new approach in derivatives pricing is promising, as it does not require aggressive assumptions of complete markets; it is logically consistent; and more importantly, it tells you, in a systematic way, what to do when market prices and your model fair values do not agree.

VIII Conclusion

The simple jump to default model is the simplest model I can think of that is both incomplete and has a continuous trading component (the stock) for dynamic hedging. Despite its simplicity, the model exhibits a rich variety of features. The new results on this very old model are made possible by the idea of embedding derivatives pricing in the context of portfolio optimization. I hope the analysis of this model has dispelled any doubt you may have on the validity and usefulness of the new approach.

Appendix: Derivation of the SJD-DOPE

I now go through a set of well-defined steps (see the recipe in Section 3.5 of [16]) to derive the SJD-DOPE. The exogenously specified dynamical equation for the stock price is

$$ds = s[(\nu - \eta\zeta)\, dt + \sigma\, dB] + \Delta s\, dQ \tag{31}$$

where $\Delta s := \eta s$ with η being the percentage jump size when the Poisson event dQ happens. In the jump to default model, η will be set to -1 in the end. But there is no harm in leaving a general η during the derivation. Applying Ito's lemma to the fair value $f^i(t,s)$ of the ith European derivative instrument leads to

$$df^i = [f^i_t + (\nu - \eta\zeta) s f^i_s + \frac{1}{2}\sigma^2 s^2 f^i_{ss}]\, dt + \sigma s f^i_s\, dB + \Delta f^i\, dQ \tag{32}$$

where $\Delta f^i := f^i(t, s + \Delta s) - f^i(t,s)$.

I introduce a shorthand notation, which will be used in the sequel. An Ito type stochastic differential equation for a dynamic state variable z is written as

$$dz = C_{(z,t)}\ dt + C_{(z,x)}\ dB^x + C_{(z,y)}\ dB^y + \cdots \tag{33}$$

where dB^x and dB^y are Brownian motions for the random factors x and y, respectively. Equation (33) should be viewed as the definition for the shorthand notation $C_{(*,*)}$ of the coefficients in a stochastic differential equation. For example, from (31) $C_{(s,t)} = (\nu - \eta\zeta)s$, and from (32) $C_{(f^i,s)} = \sigma s f^i_s$.

With n_0 shares of the stock and n_i of the ith European style derivative in the portfolio, the change of wealth dw during a small time interval dt is

$$\begin{aligned} dw &= n_0\, ds + \sum_{i=1}^{N} n_i\, df^i \\ &= \left[n_0 C_{(s,t)} + \sum_{i=1}^{N} n_i C_{(f^i,t)} \right] dt + \left[n_0 C_{(s,s)} + \sum_{i=1}^{N} n_i C_{(f^i,s)} \right] dB + \Delta w\, dQ \\ &:= C_{(w,t)}\, dt + C_{(w,s)}\, dB + \Delta w\, dQ \end{aligned} \tag{34}$$

where $\Delta w := n_0 \Delta s + \sum_{i=1}^{N} n_i \Delta f^i$. The explicit expressions for $C_{(w,t)}$ and $C_{(w,s)}$ can be found by substituting in the coefficients $C_{(s,*)}$ and $C_{(f^i,*)}$.

The value function J for this problem is defined as

$$J(t, w, s) := \sup_{n_0, n_i} E[U(w(T))] \tag{35}$$

where U is the exponential utility function and T is the investment horizon that is assumed to be longer than any derivative maturity T_i. Assuming the position $\vec{n}$ remains optimal, the necessary condition for optimality is the HJB equation

$$\sup_{n_0, n_i} [\mathcal{L}J + \zeta \Delta J] = 0 \tag{36}$$

with $\mathcal{L}J$ being defined as

$$\begin{aligned} \mathcal{L}J := & J_t + C_{(s,t)} J_s + C_{(w,t)} J_w \\ & + \frac{1}{2} C^2_{(s,s)} J_{ss} + \frac{1}{2} C^2_{(w,s)} J_{ww} + C_{(s,s)} C_{(w,s)} J_{sw} \end{aligned} \tag{37}$$

where the subscripts on J denote partial derivatives. The quantity ΔJ in (36) is defined as

$$\Delta J := J(t, w + \Delta w, s + \Delta s) - J(t, w, s) \tag{38}$$

The expression $\sup_{n_0, n_i}$ means that the first order derivative of the HJB equation with respect to n_0 and n_i should be zero, respectively. The condition $\partial(HJB)/\partial n_0 = 0$ leads to the equation

$$C_{(s,t)} J_w + C_{(s,s)} \left[C_{(w,s)} J_{ww} + C_{(s,s)} J_{sw} \right] + \zeta \Delta s \, J_w(t, w + \Delta w, s + \Delta s) = 0 \tag{39}$$

The condition $\partial(HJB)/\partial n_i = 0$ gives rise to the equation

$$C_{(f^i,t)} J_w + C_{(f^i,s)} \left[C_{(w,s)} J_{ww} + C_{(s,s)} J_{sw} \right] + \zeta \Delta f^i \, J_w(t, w + \Delta w, s + \Delta s) = 0 \tag{40}$$

For the exponential utility function, try the solution ansatz

$$J(t, w, s) = -\frac{1}{\gamma} \exp \left[-\gamma w - \gamma \bar{\phi}(t) - \gamma h(t, s) + \gamma \sum_{i=1}^{N} n_i f^i(t, s) \right] \tag{41}$$

Substituting (41) into (39) leads to

$$(\nu - \eta \zeta) - \gamma s (\hat{n}_0 + h_s) \sigma^2 + \eta \zeta \exp(-\eta \gamma \hat{n}_0 s - \gamma \Delta h) = 0 \tag{42}$$

where $\hat{n}_0$ is the optimal number of shares of the stock to hold. Recall that when there is no derivative position ($\vec{n} = \vec{0}$ and $h(t, s) = 0$), the optimal number of shares in the portfolio is the solution of the pure stock investment problem, which is denoted as $\bar{n}_0$ that satisfies

$$(\nu - \eta \zeta) - \gamma s \bar{n}_0 \sigma^2 + \eta \zeta \exp(-\eta \gamma \bar{n}_0 s) = 0 \tag{43}$$

By definition, the optimal number of hedging shares n_0^h for the derivative position is $n_0^h := \hat{n}_0 - \bar{n}_0$. From (42) and (43), n_0^h satisfies

$$\gamma s (n_0^h + h_s) \sigma^2 - \eta \zeta \exp(-\eta \gamma \bar{\pi}) \left[\exp(-\eta \gamma s n_0^h - \gamma \Delta h) - 1 \right] = 0 \tag{44}$$

where $\bar{\pi} := \bar{n}_0 s$ is the optimal directional bet amount. After setting η to minus one, (43) becomes the optimal directional bet equation (8), and (44) becomes the optimal hedging equation (12).

Substituting (41) into (40) leads to

$$f_t^i + [(\nu - \eta\zeta) - \gamma s(n_0 + h_s)\sigma^2] s f_s^i + \frac{1}{2}\sigma^2 s^2 f_{ss}^i + \zeta \exp(-\eta\gamma s n_0 - \gamma\Delta h)\, \Delta f^i = 0 \tag{45}$$

When dynamic hedging is allowed, which is the case here, n_0 takes on its optimal value $\hat{n}_0$. Under optimal dynamic hedging, the equation for f^i becomes

$$f_t^i + \frac{1}{2}\sigma^2 s^2 f_{ss}^i + \left[\frac{1}{\eta}(n_0^h + h_s)\sigma^2 + \zeta \exp(-\eta\gamma\bar{\pi})\right] (\Delta f^i - \eta s f_s^i) = 0 \tag{46}$$

This becomes the fair value linear PDE (13) after setting η to minus one.

Substituting (41) into (36) leads to (after some algebra)

$$\begin{aligned} &\bar{\phi}_t + h_t + (\nu - \eta\zeta) s h_s + \frac{1}{2}\sigma^2 s^2 h_{ss} + (\nu - \eta\zeta) s n_0 - \frac{1}{2}\gamma s^2 (n_0 + h_s)^2 \sigma^2 \\ &\quad - \frac{1}{\gamma}\zeta \left[\exp(-\eta\gamma s n_0 - \gamma\Delta h) - 1\right] + \sum_{i=1}^{N} n_i f^i(T_i, s)\delta(t - T_i) = 0 \end{aligned} \tag{47}$$

where $\delta()$ is the well-known Dirac delta function, and $f^i(T_i, s)$ is the known payoff function of the ith European derivative. To explain the source of the last term (sum of delta functions) on the left-hand side of (47), I need to go back to expression (41) for J. The term $\sum_{i=1}^{N} n_i f^i$ jumps across a maturity time T_i, because n_i jumps to zero as time goes from T_{i-} to T_{i+}. Since the value function J is continuous, h must also jump across a maturity time T_i to counteract. The delta function impulse term in (47) is needed to take into account this temporal discontinuity of h. The term $\bar{\phi}$ in (47) is the CEPL for the pure stock investment problem (*i.e.*, $\vec{n} = \vec{0}$ and $h = 0$) that satisfies the equation

$$\bar{\phi}_t + (\nu - \eta\zeta) s \bar{n}_0 - \frac{1}{2}\gamma s^2 \bar{n}_0^2 \sigma^2 - \frac{1}{\gamma}\zeta \left[\exp(-\eta\gamma s \bar{n}_0) - 1\right] = 0 \tag{48}$$

which is the same as (9) bearing in mind $\eta = -1$ and $s\bar{n}_0 = \bar{\pi}$. Under optimal dynamic hedging, n_0 takes on its optimal value $\hat{n}_0 = \bar{n}_0 + n_0^h$. Substituting (43), (44) and (48) into (47) leads to (again after some algebra)

$$\begin{aligned} &h_t + \frac{1}{2}\sigma^2 s^2 h_{ss} - \frac{1}{2}\gamma\sigma^2 s^2 (n_0^h + h_s)^2 \\ &\quad - \frac{1}{\eta}[\sigma^2 + \eta^2 \zeta \exp(-\eta\gamma\bar{\pi})] s (n_0^h + h_s) + \sum_{i=1}^{N} n_i f^i(T_i, s)\delta(t - T_i) = 0 \end{aligned} \tag{49}$$

which becomes (11) after setting η to minus one.

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